Faculty of Science, Technology, Engineering and Mathematics
M208 Pure mathematics

M208

TMA 02 2019J

Covers Book B **Cut-off date 5 December 2019**

You can submit your TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on the TMA, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' tab of the M208 website.

In the wording of the questions:

- *write down*, *list* or *state* means 'write down without justification' (unless otherwise stated)
- *find*, *determine*, *calculate*, *derive*, *evaluate* or *solve* means 'show all your working'
- *prove*, *show*, *deduce* or *verify* means 'justify each step'
- *sketch* means 'sketch without justification' and *describe* means 'describe without justification' (both unless otherwise stated).

In particular, if you use a definition, result or theorem to go from one line to the next, then you must state clearly which fact you are using – for example, you could quote the relevant unit and page, or give a Handbook reference. Remember that when you use a theorem, you must demonstrate that all the conditions of the theorem are satisfied.

The number of marks assigned to each part of a question is given in the right-hand margin, to give you a rough indication of the amount of time that you should spend on each part.

Your work should be written in a good mathematical style, as demonstrated by the exercise and worked exercise solutions in the study texts. You should explain your solutions carefully, using appropriate notation and terminology, defining any symbols that you introduce, and writing in proper sentences. Five marks (referred to as good mathematical communication, or GMC, marks) on this TMA are allocated for how well you do this.

Your score out of 5 for GMC will be recorded against Question 7. (You do not have to submit any work for this particular question.)

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3.1

You should read the information on the front page of this booklet before you start working on the questions.

Question 1 (Unit B1) – 23 marks

For each of the following sets, with the binary operation given, determine whether or not it forms a group by checking the group axioms.

(a) $(\{2, 4, 8, 16, 22, 32\}, \times_{42})$ [*8*]

(b) $(\mathbb{Z}, \circ)$, where $x \circ y = 3x + y$ [*3*]

(c) $(\mathbb{Q}, \circ)$, where $x \circ y = \frac{3}{4} + x + y$ [*12*]

Question 2 (Unit B2) – 15 marks

(a) Each of the following is a group. Determine which of these groups are cyclic. For each group that is cyclic, write down all its generators.

(i) $(G, *)$ given by the following group table.

$*$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	a	f	d	e
c	c	a	b	e	f	d
d	d	e	f	a	b	c
e	e	f	d	c	a	b
f	f	d	e	b	c	a

(ii) $(\{1, 3, 4, 9, 10, 12\}, \times_{13})$

(iii) $(H, \circ)$ given by the following group table.

$\circ$	e	v	w	x	y	z
e	e	v	w	x	y	z
v	v	y	x	z	e	w
w	w	x	y	e	z	v
x	x	z	e	v	w	y
y	y	e	z	w	v	x
z	z	w	v	y	x	e

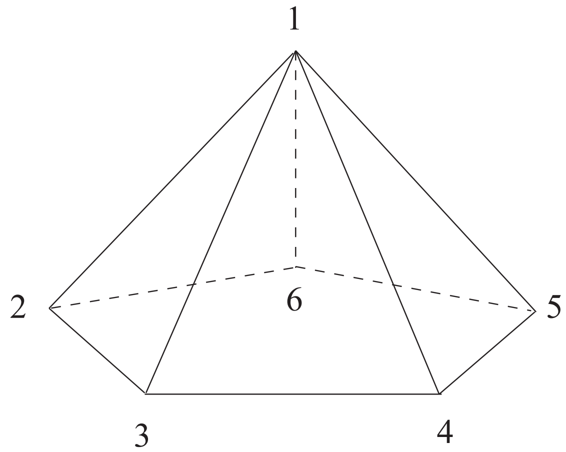
[*9*]

(b) Find an isomorphism between two of the groups listed in part (a). [*3*]

(c) Write down the order of each element of $(H, \circ)$. Determine all the distinct subgroups of $(H, \circ)$, justifying your answer. [*3*]

Question 3 (Units B1, B2 and B3) – 18 marks

The figure F, shown below, is a pyramid with a regular pentagon as its base. All the edges of F are equal in length.



(a) Write down, in cycle form, the elements of the symmetry group $S(F)$. Describe each symmetry geometrically. [8]

(b) Write down the elements of $S^{+}(F)$, the group of direct symmetries of F. Denote each of these elements by a single letter (making it clear which letter denotes which symmetry), and write down a group table for $S^{+}(F)$ using these letters. [6]

(c) State to which one of the following groups $S(F)$ is isomorphic, and show that $S(F)$ is not isomorphic to either of the other two groups:

$$(\mathbb{Z}_5, +_5), \quad (\mathbb{Z}_{10}, +_{10}), \quad S(\pentagon).$$ [4]

Question 4 (Unit B3) – 20 marks

The two-line forms of permutations p and q in S_5 are

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{pmatrix}.$$

(a) Write down the permutations p and q in cycle form. [2]

(b) Determine the permutations $p \circ q$ and $q \circ p$ in cycle form, and write down the orders of p, q, $p \circ q$ and $q \circ p$. [5]

(c) Write p and q as composites of transpositions, and state the parity of p and of q. [3]

(d) Let r be the permutation in S_5 whose cycle form is $(2\ 3\ 4)$. Find, in cycle form, the permutation $r \circ p \circ r^{-1}$; that is, find the conjugate of p by r. [2]

(e) Determine, in cycle form, the elements of the cyclic subgroup H of S_5 generated by p. [4]

(f) Determine, in cycle form, the elements of the conjugate subgroup rHr^{-1}. [4]

Question 5 (Units B1 and B4) – 8 marks

Consider the following statement.

> **Statement**
>
> The set of irrational numbers is a group under multiplication.

(a) Show that the statement is false. [2]

(b) Explain why the argument below is not a correct proof of the statement, identifying at least three errors. (There may be more than three errors, but you are required to identify only three. Your three errors should not include incorrect statements or omissions that follow entirely sensibly from earlier errors.) [6]

> **Proof (incorrect!)**
>
> We consider the group of irrational numbers under multiplication, and prove that the four group axioms hold.
>
> **G1 Closure**
>
> The set is closed under multiplication since, for example, if $x = \sqrt{2}$ and $y = \sqrt{3}$, then x and y are irrational and their product, $x \times y = \sqrt{6}$, is also irrational.
>
> **G2 Associativity**
>
> Multiplication of real numbers is commutative so it must be associative. As the set of irrational numbers is a subset of the set of real numbers, multiplication of irrational numbers will also be associative.
>
> **G3 Identity**
>
> For every irrational number x,
>
> $$x \times 1 = x = 1 \times x,$$
>
> and hence 1 is an identity element for multiplication on the irrational numbers.
>
> **G4 Inverses**
>
> For each irrational number x, the number $1/x$ exists and
>
> $$\frac{1}{x} \times x = 1 = x \times \frac{1}{x},$$
>
> so $1/x$ is an inverse of x. Thus each irrational number has an inverse with respect to multiplication.
>
> Thus the set of irrational numbers is a group under multiplication.

Question 6 (Unit B4) – 11 marks

Let G be a group and let $x, y, z \in G$.

(a) Show that the following statement is true:

$$zxz = zyz \implies x = y. \qquad [5]$$

(b) Show that the following statement need not be true:

$$xzx = yzy \implies x = y.$$

Hint: Consider the symmetry group of the rectangle. [6]

Question 7 (Book B) – 5 marks

Five marks on this assignment are allocated for good mathematical communication in your answers to Questions 1 to 6.

You do not have to submit any extra work for Question 7, but you should check through your assignment carefully, making sure that you have explained your reasoning clearly, used notation correctly and written in proper sentences. [5]